Lecture 16

Waves in Layered Media

Waves in layered media is an important topic in electromagnetics. Many media can be approximated by planarly layered media. For instance, the propagation of radio wave on the earth surface was of interest and first tackled by Sommerfeld in 1909 [121]. The earth can be approximated by planarly layered media to capture the important physics behind the wave propagation. For instance, many geophysics problems can be understood by studying waves in layered media. Many microwave components are made by planarly layered structures such as microstrip and coplanar waveguides. Layered media are also important in optics: they can be used to make optical filters such as Fabry-Perot filters. As technologies and fabrication techniques become better, there is an increasing need to understand the interaction of waves with layered structures or laminated materials.

16.1 Waves in Layered Media

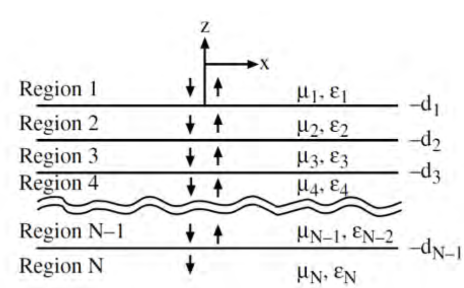

Figure 16.1: Waves in layered media. A wave entering the medium from above can be multiply reflected before emerging from the top again or transmitted to the bottom-most medium.

16.1.1 Generalized Reflection Coefficient for Layered Media

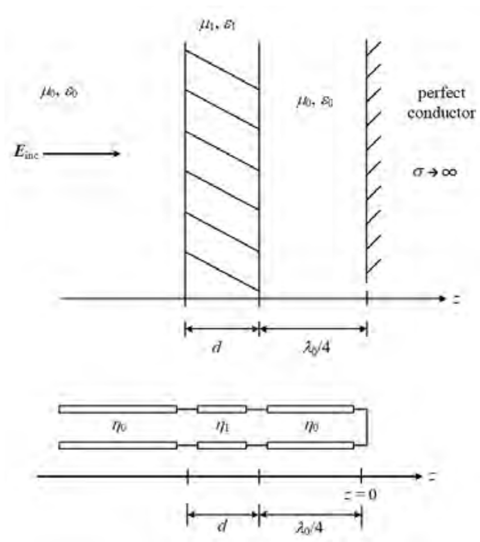

Figure 16.2: The equivalence of a layered medium problem to a transmission line problem. This equivalence is possible even for oblique incidence. For normal incidence, the wave impedance becomes intrinsic impedances (courtesy of J.A. Kong, Electromagnetic Wave Theory).

Because of the homomorphism between the transmission line problem and the plane-wave reflection by interfaces, we will exploit the simplicity of the transmission line theory to arrive at formulas for plane wave reflection by layered media. We can capitalize on using the multi-section transmission line formulas for generalized reflection coefficient, which is

$$\tilde{\Gamma}_{12} = \frac{\Gamma_{12} + \tilde{\Gamma}_{23} e^{-2j\beta_2 l_2}}{1 + \Gamma_{12}\tilde{\Gamma}_{23} e^{-2j\beta_2 l_2}} \tag{16.1.1}$$

In the above, Γ_{12} is the local reflection at the 1,2 junction, whereas $\tilde{\Gamma}_{ij}$ are the generalized reflection coefficient at the i, j interface. For instance, $\tilde{\Gamma}_{12}$ includes multiple reflections from behind the 1,2 junction. It can be used to study electromagnetic waves in layered media shown in Figures 16.1 and 16.2.

Using the result from the multi-junction transmission line, by analogy we can write down the generalized reflection coefficient for a layered medium with an incident wave at the 1,2 interface, including multiple reflections from behind the interface. We do the following replacements: $\Gamma_{12} \to R_{12}$, $\tilde{\Gamma}_{23} \to \tilde{R}_{23}$, $\tilde{\Gamma}_{12} \to \tilde{R}_{12}$, and $\beta_2 \to \beta_{2z}$. Then we have

$$\tilde{R}_{12} = \frac{R_{12} + \tilde{R}_{23} e^{-2j\beta_{2z} l_2}}{1 + R_{12}\tilde{R}_{23} e^{-2j\beta_{2z} l_2}} \tag{16.1.2}$$

where R_{12} is the local Fresnel reflection coefficient and $\tilde{R}_{ij}$ is the generalized reflection coefficient at the i, j interface. Here, l_2 is now the thickness of the region 2. In the above, we assume that the wave is incident from medium (region) 1 which is semi-infinite, the generalized reflection coefficient $\tilde{R}_{12}$ above is defined at the media 1 and 2 interface. It is assumed that there are multiple reflections coming from the right of the 2,3 interface, so that the 2,3 reflection coefficient is the generalized reflection coefficient $\tilde{R}_{23}$.

Figure 16.2 shows the case of a normally incident wave into a layered media. For this case, the wave impedance becomes the intrinsic impedance of homogeneous space.

16.1.2 Ray Series Interpretation of Generalized Reflection Coefficient

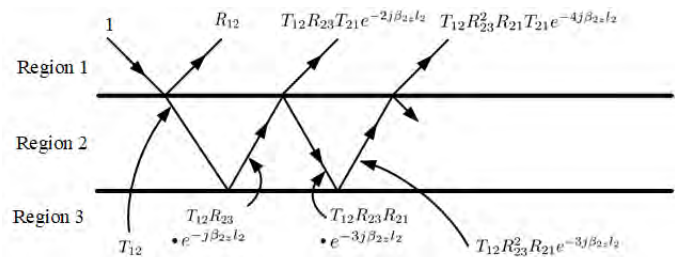

Figure 16.3: The expression of the generalized reflection coefficient into a ray series. Here, $l_2 = d_2 - d_1$ is the thickness of the slab (courtesy of [122]).

For simplicity, we will assume that $\tilde{R}_{23} = R_{23}$ in this section. By manipulation, one can convert the generalized reflection coefficient $\tilde{R}_{12}$ into a form that has a ray physics interpretation. By adding and subtracting the term

$$R_{12}^2 R_{23} e^{-2j\beta_{2z}l_2}$$

on the numerator of (16.1.2), and rearranging terms, it can be shown to become

$$\tilde{R}_{12} = R_{12} + \frac{R_{23}e^{-2j\beta_{2z}l_2}(1 - R_{12}^2)}{1 + R_{12}R_{23}e^{-2j\beta_{2z}l_2}} \tag{16.1.3}$$

By using the fact that $R_{12} = -R_{21}$ and that $T_{ij} = 1 + R_{ij}$, the above can be rewritten as

$$\tilde{R}_{12} = R_{12} + \frac{T_{12}T_{21}R_{23}e^{-2j\beta_{2z}l_2}}{1 + R_{12}R_{23}e^{-2j\beta_{2z}l_2}} \tag{16.1.4}$$

Then using the fact that $(1 - x)^{-1} = 1 + x + x^2 + \ldots +$, the above can be rewritten as

$$\tilde{R}_{12} = R_{12} + T_{12}R_{23}T_{21}e^{-2j\beta_{2z}l_2} + T_{12}R_{23}^2R_{21}T_{21}e^{-4j\beta_{2z}l_2} + \cdots . \tag{16.1.5}$$

The above allows us to elucidate the physics of each of the terms. The first term in the above is just the result of a single reflection off the first interface. The n-th term above is the consequence of the n-th reflection from the three-layer medium (see Figure 16.3). Hence, the expansion of (16.1.2) into (16.1.5) renders a lucid physical interpretation for the generalized reflection coefficient. Consequently, the series in (16.1.5) can be thought of as a *ray* series or a *geometrical optics* series. It is the consequence of multiple reflections and transmissions in region 2 of the three-layer medium. It is also the consequence of expanding the denominator of the second term in (16.1.4). Hence, the denominator of the second term in (16.1.4) can be physically interpreted as a consequence of multiple reflections within region 2.

16.2 Phase Velocity and Group Velocity

Now that we know how a medium can be frequency dispersive in a complicated fashion as in the Drude-Lorentz-Sommerfeld (DLS) model, we are ready to investigate the difference between the phase velocity and the group velocity. In this course, we will use k and β interchangeably to represent wavenumber.

16.2.1 Phase Velocity

The phase velocity is the velocity of the phase of a wave. It is only defined for a monochromatic signal (also called time-harmonic, CW (constant wave), or sinusoidal signal) at one given frequency. Given a sinusoidal wave signal, e.g., the voltage signal on a transmission line, using phasor technique, its representation in the time domain can be easily found and take the form

$$\begin{aligned} V(z,t) &= V_0 \cos(\omega t - kz + \alpha) \\ &= V_0 \cos\left[k\left(\frac{\omega}{k}t - z\right) + \alpha\right] \end{aligned} \tag{16.2.1}$$

This sinusoidal signal moves with a velocity

$$v_{ph} = \frac{\omega}{k} \tag{16.2.2}$$

where, for example, $k = \omega\sqrt{\mu\varepsilon}$, inside a simple coax. Hence,

$$v_{ph} = 1/\sqrt{\mu\varepsilon} \tag{16.2.3}$$

But a dielectric medium can be frequency dispersive, or $\varepsilon(\omega)$ is not a constant but a function of ω as has been shown with the Drude-Lorentz-Sommerfeld model. Therefore, signals with different ω's will travel with different phase velocities.

More bizarre still, what if the coax is filled with a plasma medium where

$$\varepsilon = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2}\right) \tag{16.2.4}$$

Then, $\varepsilon < \varepsilon_0$ always meaning that the phase velocity given by (16.2.3) can be larger than the velocity of light in vacuum (assuming $\mu = \mu_0$). Also, $\varepsilon = 0$ when $\omega = \omega_p$, implying that $k = 0$; then in accordance to (16.2.2), $v_{ph} = \infty$. These ludicrous observations can be justified or understood only if we can show that information can only be sent by using a wave packet.[1] The same goes for energy which can only be sent by wave packets, but not by CW signal; only in this manner can a finite amount of energy be sent. Therefore, it is prudent for us to study the velocity of a wave packet which is not a mono-chromatic signal. These wave packets can only travel at the group velocity as shall be shown, which is always less than the velocity of light.

16.2.2 Group Velocity

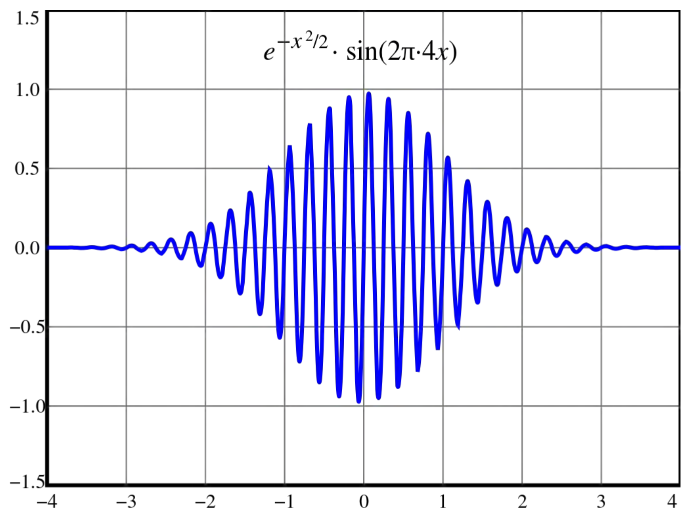

Figure 16.4: A Gaussian wave packet can be thought of as a linear superposition of monochromatic waves of slightly different frequencies. If one Fourier transforms the above signal, it will be a narrow-band signal centered about certain ω_0 (courtesy of Wikimedia [123]).

Now, consider a narrow band wave packet as shown in Figure 16.4. It cannot be mono-chromatic, but can be written as a linear superposition of many frequencies. One way to express this is to write this wave packet as an integral in terms of Fourier transform, or a summation over many frequencies, namely[2]

$$V(z,t) = \int_{-\infty}^{\infty} d\omega \underset{\sim}{V}(z,\omega) e^{j\omega t} \tag{16.2.5}$$

[1] In information theory, according to Shannon, the basic unit of information is a bit, which can only be sent by a digital signal, or a wave packet.

[2] The Fourier transform technique is akin to the phasor technique, but different. For simplicity, we will use $\underset{\sim}{V}(z,\omega)$ to represent the Fourier transform of $V(z,t)$.

To make $V(z,t)$ be related to a traveling wave, we assume that $\underset{\sim}{V}(z,\omega)$ is the solution to the one-dimensional Helmholtz equation[3]

$$\frac{d^2}{dz^2}\underset{\sim}{V}(z,\omega) + k^2(\omega)\underset{\sim}{V}(z,\omega) = 0 \tag{16.2.6}$$

To derive this equation, one can easily extend the derivation in Section 7.2 to a dispersive medium where $V(z,\omega) = E_x(z,\omega)$. Alternatively, one can generalize the derivation in Section 11.2 to the case of dispersive transmission lines. For instance, when the co-axial transmission line is filled with a dispersive material, then $k^2 = \omega^2\mu_0\varepsilon(\omega)$. Thus, upon solving the above equation, one obtains that $V(z,\omega) = V_0(\omega)e^{-jkz}$, and

$$V(z,t) = \int_{-\infty}^{\infty} d\omega V_0(\omega)e^{j(\omega t - kz)} \tag{16.2.7}$$

In the above, $V(z,t)$ is real value. As such, the negative frequency components of the above integral have to be complex conjugate of the positive frequency components. We can also rewrite the above as

$$V(z,t) = \int_{-\infty}^{0} d\omega V_0(\omega)e^{j(\omega t - kz)} + \int_{0}^{\infty} d\omega V_0(\omega)e^{j(\omega t - kz)} \tag{16.2.8}$$

Using the fact that $V_0(-\omega) = V_0^*(\omega)$ and that $k(-\omega) = k^*(\omega)$, we can write the above as sum over only the $+\omega$ part of the integral and take twice the real part of the integral.

$$V(z,t) = 2\Re e \int_{0}^{\infty} d\omega V_0(\omega)e^{j(\omega t - kz)} \tag{16.2.9}$$

In the general case, k is a complicated function of ω as shown in Figure 16.5.

[3]In this notes, we will use k and β interchangeably for wavenumber. The transmission line community tends to use β while the optics community uses k.

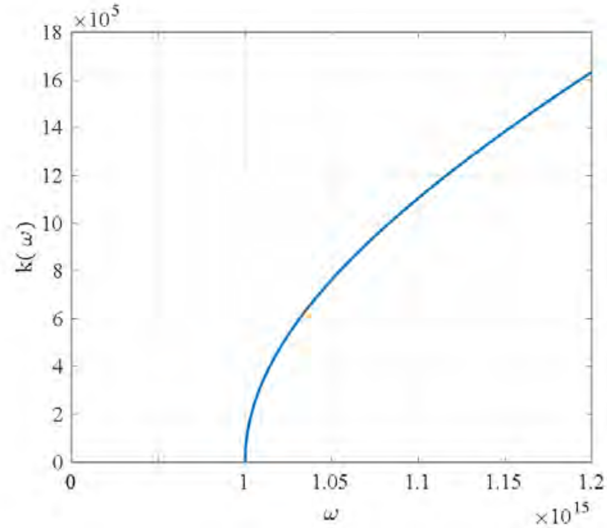

Figure 16.5: A typical frequency dependent $k(\omega)$ albeit the frequency dependence can be more complicated than shown here.

Since this is a wave packet, we assume that $V_0(\omega)$ is narrow band centered about a frequency ω_0, the carrier frequency as shown in Figure 16.6. Therefore, when the integral in (16.2.7) is performed, we need only sum over a narrow range of frequencies in the vicinity of ω_0.

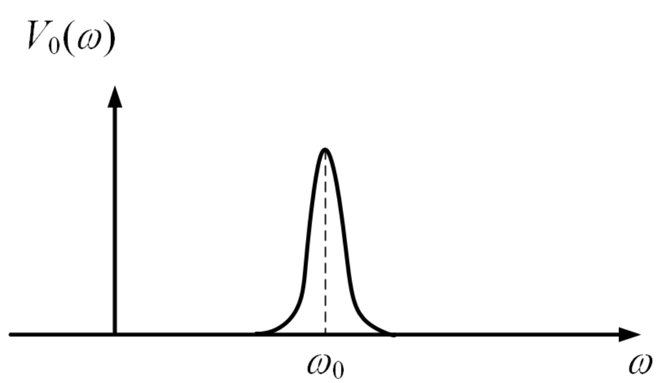

Figure 16.6: The frequency spectrum of $V_0(\omega)$ which is the Fourier transform of $V_0(t)$.

Henceforth, we can approximate the integrand in the vicinity of $\omega = \omega_0$, in particular,

$k(\omega)$ by Taylor series expansion, and let

$$k(\omega) \cong k(\omega_0) + (\omega - \omega_0)\frac{dk(\omega_0)}{d\omega} + \frac{1}{2}(\omega - \omega_0)^2\frac{d^2k(\omega_0)}{d\omega^2} + \cdots \tag{16.2.10}$$

Since we need to integrate over $\omega \approx \omega_0$, we can substitute (16.2.10) into (16.2.9) and rewrite it as

$$V(z,t) \cong 2\Re e\left[e^{j[\omega_0 t - k(\omega_0)z]}\underbrace{\int\limits_0^\infty d\omega V_0(\omega)e^{j(\omega-\omega_0)t}e^{-j(\omega-\omega_0)\frac{dk}{d\omega}z}}_{F\left(t-\frac{dk}{d\omega}z\right)}\right] \tag{16.2.11}$$

where more specifically,

$$F\left(t - \frac{dk}{d\omega}z\right) = \int\limits_0^\infty d\omega V_0(\omega)e^{j(\omega-\omega_0)t}e^{-j(\omega-\omega_0)\frac{dk}{d\omega}z} \tag{16.2.12}$$

It can be seen that the above integral now involves the integral summation over a small range of ω in the vicinity of ω_0. By a change of variable by letting $\Omega = \omega - \omega_0$, it becomes

$$F\left(t - \frac{dk}{d\omega}z\right) = \int_{-\Delta}^{+\Delta} d\Omega V_0(\Omega + \omega_0)e^{j\Omega\left(t-\frac{dk}{d\omega}z\right)} \tag{16.2.13}$$

When Ω ranges from $-\Delta$ to $+\Delta$ in the above integral, the value of ω ranges from $\omega_0 - \Delta$ to $\omega_0 + \Delta$. It is assumed that outside this range of ω, $V_0(\omega)$ is sufficiently small so that its value can be ignored.

The above itself is a Fourier transform integral that involves only the low frequencies of the Fourier spectrum where $e^{j\Omega\left(t-\frac{dk}{d\omega}z\right)}$ is evaluated over small Ω values. Hence, F is a slowly varying function. Moreover, this function F moves with a velocity

$$v_g = \frac{d\omega}{dk} \tag{16.2.14}$$

Here, $F(t-\frac{z}{v_g})$ in fact is the velocity of the envelope in Figure 16.4. In (16.2.11), the envelope function $F(t-\frac{z}{v_g})$ is multiplied by the rapidly varying function

$$e^{j[\omega_0 t - k(\omega_0)z]} \tag{16.2.15}$$

before one takes the real part of the entire function. Hence, this rapidly varying part represents the rapidly varying carrier frequency shown in Figure 16.4. More importantly, this carrier, the rapidly varying part of the signal, moves with the velocity

$$v_{ph} = \frac{\omega_0}{k(\omega_0)} \tag{16.2.16}$$

which is the phase velocity.

16.3 Wave Guidance in a Layered Media

Now that we have understood phase and group velocity, we are at ease with studying the propagation of a guided wave in a layered medium. We have seen that in the case of a surface plasmonic resonance, the wave is guided by an interface because the Fresnel reflection coefficient becomes infinite. This physically means that a reflected wave exists even if an incident wave is absent or vanishingly small. This condition can be used to find a guided mode in a layered medium, namely, to find the condition under which the generalized reflection coefficient (16.1.2) will become infinite.[4]

16.3.1 Transverse Resonance Condition

Therefore, to have a guided mode exist in a layered medium due to multiple bounces, the generalized reflection coefficient becomes infinite, the denominator of (16.1.2) is zero, or that

$$1 + R_{12}\tilde{R}_{23}e^{-2j\beta_{2z}l_2} = 0 \tag{16.3.1}$$

where t is the thickness of the dielectric slab. Since $R_{12} = -R_{21}$, the above can be written as

$$1 = R_{21}\tilde{R}_{23}e^{-2j\beta_{2z}l_2} \tag{16.3.2}$$

The above has the physical meaning that the wave, after going through two reflections at the two interfaces, 21, and 23 interfaces, which are R_{21} and $\tilde{R}_{23}$, plus a phase delay given by $e^{-2j\beta_{2z}l_2}$, becomes itself again. This is also known as the transverse resonance condition. When specialized to the case of a dielectric slab with two interfaces and three regions, the above becomes

$$1 = R_{21}R_{23}e^{-2j\beta_{2z}l_2} \tag{16.3.3}$$

The above can be generalized to finding the guided mode in a general layered medium. It can also be specialized to finding the guided mode of a dielectric slab.

[4]As mentioned previously in Section 15.1.1, this is equivalent to finding a solution to a problem with no driving term (forcing function), or finding the homogeneous solution to an ordinary differential equation or partial differential equation. It is also equivalent to finding the null space solution of a matrix equation.